MONOTONICITY OF EFFECTIVE ADMITTANCE OF A NETWORK

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Abstract. This paper examines the effective admittance of an electrical network, using a weighted graph as its representation. The resulting method of calculation is to use a formula belonging to Haynsworth to compute the effective admittance. We show that this value is also positive. Also, the derivative of the quadratic form, representing power dissipated, is implemented to determine how various changes to the graph affect the effective admittance.

Contents

1. Introduction

To begin with, we will be looking only at connected networks. One important characteristic of these electrical networks is that there is a distinction between boundary and interior nodes [1]. Our networks may be complicated and certainly nonplanar, so it is efficient to have a matrix formula to calculate the effective admittance. The matrices involved are part of an electrical network problem that may be stated as an inverse problem or a forward problem, depending on the information available [1].

The information in these problems include the structure of the graph, the admittances on each edge, voltage applied at the boundary nodes, and the currents at boundary nodes. The method of relation between the different data is to use Kirchhoff's current and voltage laws. The formula to calculate the effective admittance is used as another tool for solving

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these problems. It also allows for a derivative to be taken in order to better understand the result of changes to the network.

2. The Network

We begin with a graph, Ω, with its boundary denoted as $\partial\Omega$. Our network is made up of voltage sources and electrical elements that are reactive, conductive, or a mixture of both. The admittances for these elements are given on the edges of the graph and the voltage is measured between two nodes in Ω. A known voltage source is applied to each of the boundary nodes. The resulting current is measured at the boundary nodes, with positive current flowing into the network. By Kirchhoff's current law, the net current through any interior node is zero.

2.1. **The Effective Admittance.** The effective admittance is equal to the admittance measured between two nodes. Physically, this would be measured with an ohmmeter, which applies a potential difference between the two nodes and measures the resulting current. With these the current and voltage known, the conductance can be calculated using Ohm's law. Simply, the effective admittance is the value of a component that could replace everything between those two nodes. The value of such an admittance should be positive for a physical case, and this will be explicitly shown.

In our graphs, the boundary nodes are denoted by a closed circle. The interior nodes are denoted by an open circle. Conductances are given on non-directional edges. Below is an example, where the effective admittance would be taken between nodes 1 and 2. The admittances are represented by α, β, θ, τ, and ρ. They combine in particular ways to contribute to the overall conductance.

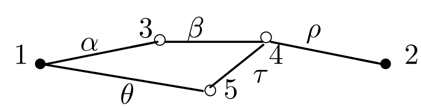


Figure 1. This is a complete graph, Ω, with boundary nodes 1 and 2.

2.2. **Matrix Representation.** The most important mathematical relationship in our network is given by Ohm's Law. Typically, it is expressed as $V = IR$. However, it is convenient to replace R with its inverse, C, the conductance. So, we have that $V = I/C$ or $VC = I$. This can be expressed by partitioned matrices. In particular, it becomes:

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \phi \\ 0 \end{bmatrix} \tag{1}$$

The leftmost matrix, in particular, is called a Kirchhoff matrix. From this matrix is derived a response matrix, Λ, which will defined shortly. When considering admittances, the formula for power dissipated is dependent on a frequency, ω, but taken at a particular time, it is not any different from considering only the real, power-dissipating part.

2.2.1. *The Kirchhoff Matrix.* The network itself can be expressed by a Kirchhoff matrix, K, with admittances between adjacent nodes as entries[1]. It is partitioned by node type. The partition A contains boundary-to-boundary edges, B contains interior-to-boundary edges, B^T contains boundary-to-interior edges, and C contains interior-to-interior edges.

$$K = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$$

When right-multiplied by a partitioned matrix of boundary voltages, x, and interior voltages, y, the result is a vector of currents into the network. The net current at any interior node is zero, by Kirchhoff's current law, and ϕ on the boundary nodes (1). It is a symmetric matrix with row sums and column sums equal to zero. It has positive diagonal entries and nonpositive off diagonal entries.

2.2.2. *The Response Matrix.* The current into a boundary node can also be produced by a response matrix, Λ. It maps a boundary voltage vector to a boundary current vector [1]. Like the Kirchhoff matrix, it has nonpositive off-diagonal entries and positive entries on the diagonal. By Kirchhoff's current law, the row sums and column sums are zero, and it is a symmetric matrix. In the book by Morrow and Curtis, it is given in Theorem 3.9 as the Schur complement of the Kirchhoff matrix [1].

$$\Lambda = A - BC^{-1}B^T \tag{2}$$

This formula depends on C being invertible. A proof of which is proven previously inMorrow's book. All of the other components are unaltered partitions of K. The re.sponse matrix has qualities which make it nicer to work with than K.

3. Calculating Effective Admittance

There are several approaches to calculating the effective admittance. First, one may consider the power dissipated by the entire network. If it increases, then so must the effective admittance, for a fixed set of boundary voltages. Secondly, there exists a matrix formula to calculate the effective admittance from K. We will begin with only conductances, which have no imaginary part, and then extend our method to admittances.

3.1. **Monotonicity of Effective Conductance.**

Showing that the effective conductance is monotonic is as step toward showing the effective admittance is monotonic. It is not yet clear that the matrix properties hold for complex-valued entries. The reactive portion of an element does not dissipate power, however, it does affect the phase of the power dissipated. This makes the values measured time dependent. When allowing complex-valued admittances, the power dissipated will be taken at t = 0. When increasing an edge of a graph, we will consider only increasing the real part, for the moment.

3.2. **Power Dissipated and Effective Conductance.** The objective of this research is to determine how the effective admittance depends on any given edge. But first we need to look at the effective conductance. One way to do this is to determine how much power is dissipated by an effective conductance and see how it changes when an edge is altered. The expression for power dissipated is: power = voltage × current. In terms of our matrices, with u as a voltage vector and the product of Ku computing the current, power dissipated becomes $u^T Ku$. Since the entries that alter belong the the Kirchhoff matrix, but we are just working on the boundary, it is helpful to show that, for some voltage vector $u^T = \begin{bmatrix} x & y \end{bmatrix}$, we have that $u^T Ku = x^T \Lambda x$.

Proof.

$$Ku = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \phi \\ 0 \end{bmatrix} = \begin{bmatrix} Ax + By \\ B^T x + Cy \end{bmatrix}$$

$$\phi = Ax + By$$

$$y = -C^{-1} B^T x$$

$$\phi = (A - BC^{-1} B^T)x = \Lambda x$$

$$Ku = \begin{bmatrix} \Lambda x \\ 0 \end{bmatrix}, \; and \; u^T Ku = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} \Lambda x \\ 0 \end{bmatrix} = x^T \Lambda x$$

□

With this substitution, the power dissipated may be rewritten as $x^T \Lambda x$. Now, we may prove the monotonicity of the effective admittance by showing that, for Λ_{new} with an increased edge, $x^T \Lambda_{new} x \geq x^T \Lambda_{old} x$. We will consider values on edges that are purely conductances and then extend to allow admittances.

3.3. **Increasing Edges.** The first step in examining increasing an edge is to look at the different types of edges. Let $\delta \ \epsilon \ \partial\Omega$ and $int \ \epsilon$ interior of Ω. The first thing we want to consider is if it is a δ-δ edge, δ-*int* edge, or *int*-*int* edge. It turns out that a previous calculation of the differential of L can be used to show that the power dissipated is monotonic and independent of whether the increased edge is a boundary edge or an interior-edge[1]. To do this, one must recognize the the power dissipated is also the quadratic form.

Theorem 3.1. The derivative of the quadratic form, $x^T \Lambda x$, in a connected graph, Ω, with conductances, γ, and voltages, u, is positive when an edge, or multiple edges, are increased by a value, κ, with $\kappa \geq 0$.

Proof. Replace the bilinear form $< y, \Lambda x >$ with the quadratic form $< x^T, \Lambda x >$. Let g_i be the voltage at node i and let $\nabla_{i,j} g = g_i - g_j$. The quadratic form becomes:

$$< x^T, \Lambda x > = \; Q_\gamma = \sum \gamma_{i,j} (u_i - u_j)^2 \tag{3}$$

Following the proof on pages 77-79 and substituting $\gamma_{i,j} + t\kappa$ for the current value, $\gamma_{i,j}$, and $u_t = u + \delta u_t$, for u, yields our desired result. Here κ is a change in a conductance, and δu_t is a voltage function which is zero on the boundary nodes of Ω.

In the book by Morrow and Curtis, equation 4.5 in section 4.6 shows,

$$\frac{d}{dt} B_{\gamma+t\kappa}|_{t=0} = \sum \kappa_{i,j} \nabla_{i,j} u \nabla_{i,j} w \tag{4}$$

Here we have two voltage functions u and w, with a change in conductance along edge i, j of $\kappa_{i,j}$. It follows that for the quadratic form, when $\nabla_{i,j} u = \nabla_{i,j} w$, we have

$$\frac{d}{dt} Q_{\gamma+t\kappa}|_{t=0} = \sum \kappa_{i,j} (\nabla_{i,j} u)^2 \tag{5}$$

This value is positive for any increase, κ_i, on an edge. Likewise, the derivative of the quadratic form, Q is negative for a negative change to an edge. So, we have proven that the effective conductance is monotonic. □

3.3.1. *Adding New Edges and Vertices.* Adding a new edge is comparable to increasing an edge from zero. Such an edge can be seen as a virtual edge. In any case, the same logic used for increasing an existing edge value can be applied. An interesting case that has not been explored is adding a node to either the boundary or the interior. This would require a different approach because it would change the size of the Kirchhoff matrix. This would be an interesting problem to pursue.

3.4. **Monotonicity of Effective Admittance.** It is a different problem when complex numbers are involved. It is not clear that row sums or column sums must add to zero. Other properties of the Kirchhoff and response matrix may not hold. In order to use these matrices, one must first show that the Dirichlet (forward) problem is well-posed and that it would have a unique solution for each frequency of the reactive elements. To make matters simple, this was proven in a paper by Hashemi in 2005 for a connected network with at least one resistor [2]. This makes sense: having no resistors would dissipate no power.

3.4.1. *Invertibility of C.* With the matrix K being admissible, the next step is to show that C is invertible. A proof with a minor flaw is given in the Hashemi paper, for which I provide a small correction. In the proof given, the expansion of the a quadratic form leave off terms due to the rows ad columns in C not summing to zero.

Lemma 3.1. For a graph, Ω, with admittances, γ, the partition C of the Kirchhoff matrix, K, is invertible.

Proof. Suppose that for some voltage vector, x,with frequency γ, $Cx(\gamma) = 0$. It follows that since no current flows, $\bar{x}^T Cx = 0$. As a sum, we have,

$$\begin{aligned}
\bar{x}^T Cx &= \sum_{i,j} \bar{x}_i(\omega)C(\omega)x_j(\omega) \\
&= \sum_{i\neq j} \bar{x}_i(\omega)C(\omega)x_j(\omega) + \sum_{i=1}^{n} C_{ii}(\omega)|x_i(\omega)|^2 + \sum_{i=1}^{n} d_{ii}(\omega)|x_i(\omega)|^2 \\
&= \sum_{i<j} C_{ij}(\omega)[\bar{x}_i(\omega)x_j(\omega) + \bar{x}_j(\omega)x_i(\omega)] + \sum_{i=1}^{n} C_{ii}(\omega)|x_i(\omega)|^2 + \sum_{i=1}^{n} d_{ii}(\omega)|x_i|^2 \\
&= \sum_{i<j} C_{ij}(\omega)[x_i(\omega) - x_j(\omega)][\bar{x}_j(\omega) - \bar{x}_i(\omega)] + \sum_{i=1}^{n} d_{ii}(\omega)|x_i|^2 \\
&= \sum_{i<j} -C_{ij}(\omega)|x_i(\omega) - x_j(\omega)|^2 + \sum_{i=1}^{n} d_{ii}(\omega)|x_i|^2 = \ 0
\end{aligned}$$

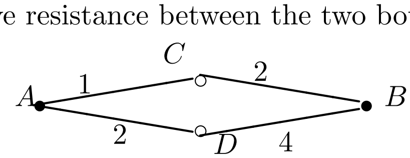
But this must mean that $x_i(\omega) = x_j(\omega)$ for all i, j, ω. Also, d_i or x_i must also be zero. If this is true then x is a constant vector. These conditions imply that some new vector u composed of x, with zeros in some entries, $u^T Ku = 0$. This implies that u is also a constant vector. If this is true, since u contains some zeros, it must be the zero vector. The conclusion is that C is invertible for all cases except in the trivial case. □

This proof is essentially the same as that presented by Hashemi. Added are the values d_{ii} that would contribute to the row sums in C being zero, since that is a criteria of using the quadratic form's expression. They do not alter the result of the original proof, but are necessary.

3.5. **Haynsworth Formula for Effective Admittance.** A simple way to calculate the effective admittance is with a modified formula developed by Emilie Haynsworth, which is based on the Schur complement [3]. It takes the Kirchhoff Matrix and uses subdeterminants It can be modified to produce a single value.

As yet, I have not considered how the formula works beyond the case with two boundary vertices. It should extend easily for more vertices and this would be an interesting problem. To begin with, you have a graph, Ω, which is represented by K. In the partition A are boundary-to-boundary edge values. For calculating the effective admittance between two boundary edges, take the a boundary node entry $a_{2,2}$, the corresponding row in B, the corresponding column in B^T, and the entirety of C. Form a new square matrix (6). Then take its determinant and divide this by the determinant of C, since we know C is invertible. This will produce the effective resistance between the two boundary nodes.

Example 3.1.



The affective admittance between A and B can be calculated by hand as : $\frac{2*4}{2+4} + \frac{2*1}{2+1} = 2$. or it can be calculated by the formula:

Take the Kirchhoff matrix, $\begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$. Take the entry $a_2 2$ and the row to the right, b, the column below it, b^T, and C to form a new matrix, $\begin{bmatrix} a_{2,2} & b \\ b^T & C \end{bmatrix}$. Now, the effective admittance between the two boundary nodes is given by:

$$\frac{det \begin{bmatrix} a_{2,2} & b \\ b^T & C \end{bmatrix}}{det[C]} = \frac{det \begin{bmatrix} 6 & -2 & -4 \\ -2 & 3 & 0 \\ -4 & 0 & 6 \end{bmatrix}}{det \begin{bmatrix} 0 & 3 \\ 0 & 6 \end{bmatrix}} = \frac{36}{18} = 2. \tag{6}$$

Since the formula works, it would be a simple way to determine if the effective admittance is monotonic. One would simply have to take the derivative. It becomes much more complicated with increase numbers of boundary vertices.

3.5.1. *Proving Effective Admittance is Monotonic.*

Theorem 3.1. The real part of the effective admittance is positive for a connected network, Ω, with conductances, γ_{ij}, when an edge, $a_{2,2}$ is increased by a complex value, t, where $\text{Re}(t) > 0$.

Proof. Suppose we have a network and we know that for every conductance, the real part is positive, which is the physical case [2]. That implies that det $\begin{bmatrix} a_{2,2} & b \\ b^T & C \end{bmatrix}$ and that det[C] $\neq 0$. Also it implies that the effective admittance, $\frac{det \begin{bmatrix} a_{2,2} & b \\ b^T & C \end{bmatrix}}{det[C]}$, is not zero. So, consider increase the edge, $a_{2,2}$, by t and producing

$$\frac{det \begin{bmatrix} a_{2,2} + t & b \\ b^T & C \end{bmatrix}}{det[C]} = \frac{det \begin{bmatrix} a_{2,2} + t & b \\ b^T & C \end{bmatrix} + det \begin{bmatrix} t & 0 \\ b^T & C \end{bmatrix}}{det[C]}$$
$$= \frac{det \begin{bmatrix} a_{2,2} & b \\ b^T & C \end{bmatrix}}{det[C]} + t$$

Suppose that this sum is zero for some choice of t. Then that would occur exactly when t=$-\frac{det \begin{bmatrix} a_{2,2} & b \\ b^T & C \end{bmatrix}}{det[C]}$. For this value of t, the $\text{Re}(t)$ cannot be > 0. This is only true if $\text{Re}(t)$ < 0, which imply that the value $-t = \frac{det \begin{bmatrix} a_{2,2} & b \\ b^T & C \end{bmatrix}}{det[C]}$ would only be able to have $\text{Re}(-t) > 0$.

So the effective admittance, $\frac{det\begin{bmatrix} a_{2,2} & b \\ b^T & C \end{bmatrix}}{det[C]}$, must have positive real part for any increased value of t.

$\square$

I would have like to have a theorem proof for any size network, but extension remains incomplete. However, thiscurrent proof does extend to increasing the value of multiple edges and possibly adding edges. It would be interesting to see that happens when values in C are changed. It is still an open question whether or not adding vertices is possible to consider in a similar manner. There are many more questions to be resolved about more general, nonphysical cases. There are also open questions about how changes to the graph affect recoverability. It would be interesting to look at how the effective resistance can add to the recovery process.

References

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